



MARKSCHEME

May 2011

FURTHER MATHEMATICS

Standard Level

Paper 2

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Instructions to Examiners

Abbreviations

- M** Marks awarded for attempting to use a correct **Method**; working must be seen.
- (M)** Marks awarded for **Method**; may be implied by **correct** subsequent working.
- A** Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding **M** marks.
- (A)** Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.
- R** Marks awarded for clear **Reasoning**.
- N** Marks awarded for **correct** answers if **no** working shown.
- AG** Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations **MI**, **AI**, etc.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award **M0** followed by **AI**, as **A** mark(s) are often dependent on the preceding **M** mark.
- Where **M** and **A** marks are noted on the same line, e.g. **MIAI**, this usually means **MI** for an **attempt** to use an appropriate method (e.g. substitution into a formula) and **AI** for using the **correct** values.
- Where the markscheme specifies **(M2)**, **N3**, etc. do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 N marks

*Award N marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of **N** and other marks.
- There may be fewer **N** marks available than the total of **M**, **A** and **R** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets e.g. (M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer **FT** marks.
- If the error leads to an inappropriate value (e.g. $\sin \theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent A** marks can be awarded, but **M** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). Apply a **MR** penalty of 1 mark to that question. Award the marks as usual and then write $-1(\mathbf{MR})$ next to the total. Subtract 1 mark from the total for the question. A candidate should be penalized only once for a particular mis-read.*

- If the question becomes much simpler because of the **MR**, then use discretion to award fewer marks.
- If the **MR** leads to an inappropriate value (e.g. $\sin \theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks (*d*)

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled (**d**) and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, etc.
- Alternative solutions for part-questions are indicated by **EITHER . . . OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates may not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x - 3)$, the markscheme gives:

$$f'(x) = (2\cos(5x - 3))5 \quad (= 10\cos(5x - 3)) \quad \mathbf{AI}$$

Award **AI** for $(2\cos(5x - 3))5$, even if $10\cos(5x - 3)$ is not seen.

10 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

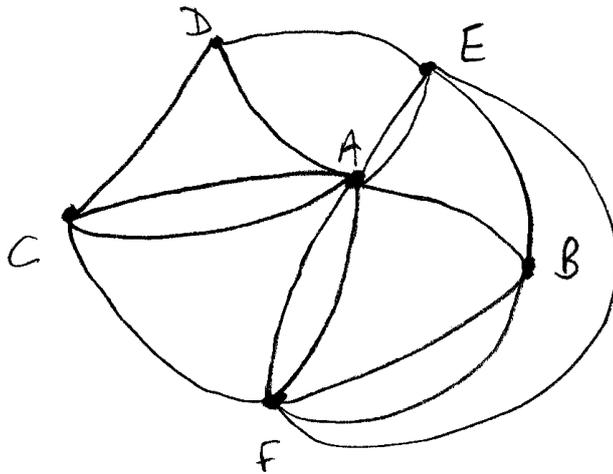
- **Rounding errors:** only applies to final answers not to intermediate steps.
- **Level of accuracy:** when this is not specified in the question the general rule applies: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Candidates should be penalized **once only IN THE PAPER** for an accuracy error (**AP**). Award the marks as usual then write (**AP**) against the answer. On the **front** cover write $-1(\mathbf{AP})$. Deduct 1 mark from the total for the paper, not the question.

- If a final correct answer is incorrectly rounded, apply the **AP**.
- If the level of accuracy is not specified in the question, apply the **AP** for correct answers not given to three significant figures.

If there is no working shown, and answers are given to the correct two significant figures, apply the **AP**. However, do **not** accept answers to one significant figure without working.

1. (a)



A2
[2 marks]

(b)

$$M = \begin{matrix} & \begin{matrix} A & B & C & D & E & F \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \end{matrix} & \begin{pmatrix} 0 & 1 & 2 & 1 & 2 & 2 \\ 1 & 0 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

A2
[2 marks]

Note: Award *A1* for one error or omission, *A0* for more than one error or omission. Two symmetrical errors count as one error.

(c)

$$\begin{matrix} A & B & C & D & E & F \\ (8, & 4, & 4, & 3, & 5, & 6) \end{matrix}$$

A2
[2 marks]

Note: Award no more than *A1* for one error, *A0* for more than one error.

(d) (i) no, because there are odd vertices M1A1

(ii) yes, because there are exactly two odd vertices M1A1

[4 marks]

continued ...

Question 1 continued

(e)

$$M^4 = \begin{matrix} & \begin{matrix} A & B & C & D & E & F \end{matrix} \\ \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \end{matrix} & \begin{pmatrix} 309 & 174 & 140 & 118 & 170 & 214 \\ 174 & 117 & 106 & 70 & 122 & 132 \\ 140 & 106 & 117 & 66 & 134 & 138 \\ 118 & 70 & 66 & 53 & 80 & 102 \\ 170 & 122 & 134 & 80 & 157 & 170 \\ 214 & 132 & 138 & 102 & 170 & 213 \end{pmatrix} \end{matrix}$$

number of walks of length 4 is 170

(M1)A1

Note: The complete matrix need not be shown. Only one of the FE has to be shown.

[2 marks]

Total [12 marks]

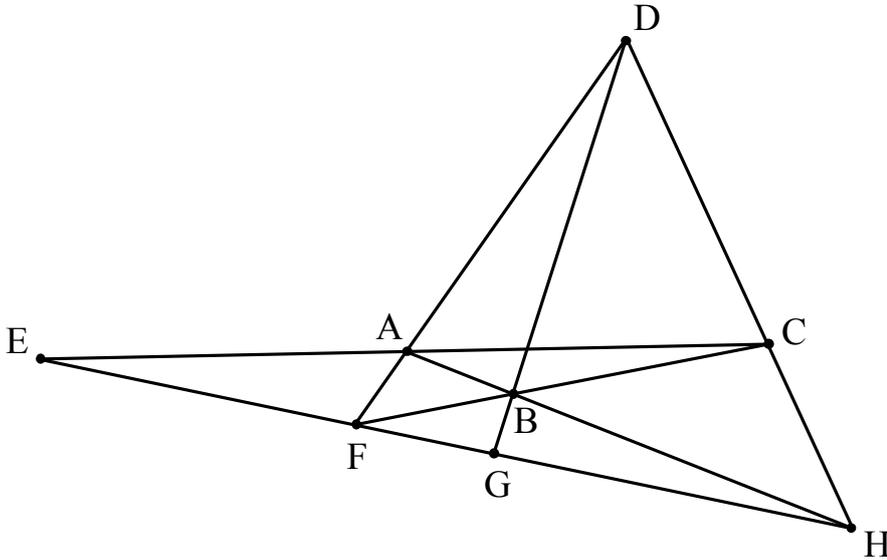
2. (a) (i) $PC_1^2 - r_1^2 = PC_2^2 - r_2^2$ M1
 $PC_1^2 = PQ^2 + C_1Q^2$
 $PC_2^2 = PQ^2 + QC_2^2$ A1
 $\Rightarrow PQ^2 + C_1Q^2 - r_1^2 = PQ^2 + QC_2^2 - r_2^2$ M1
 $\Rightarrow C_1Q^2 - QC_2^2 = r_1^2 - r_2^2$
 $\Rightarrow (C_1Q - QC_2)(C_1Q + QC_2) = r_1^2 - r_2^2$ A1
 $\Rightarrow (C_1Q - QC_2)(C_1C_2) = r_1^2 - r_2^2$ A1
 $\Rightarrow (C_1Q - QC_2) = \frac{r_1^2 - r_2^2}{C_1C_2}$ AG
- (ii) centres are $(-1, 5)$ and $(5, -3)$ A1
radii are 3 and 2 A1
 $C_1C_2 = 10$ A1
 $C_1Q - QC_2 = \frac{5}{10}$ A1
 $C_1Q + QC_2 = 10$ A1
so $C_1Q = \frac{21}{4}$ and $C_2Q = \frac{19}{4}$ A1A1
so $\frac{C_1Q}{C_2Q} = \frac{21}{19}$ AG

[12 marks]

continued ...

Question 2 continued

(b)



in $\triangle HFD$, $[HA]$, $[FC]$ and $[DG]$ are concurrent at B

MI

so, $\frac{HG}{GF} \times \frac{FA}{AD} \times \frac{DC}{CH} = 1$ by Ceva's theorem

AIRI

in $\triangle HFD$, with CAE as transversal,

MI

$\frac{HE}{EF} \times \frac{FA}{AD} \times \frac{DC}{CH} = -1$ by Menelaus' theorem

AIRI

therefore, $\frac{HG}{GF} = -\frac{HE}{EF}$

AG

[6 marks]

Total [18 marks]

3. (a) interval is $\frac{852}{1400} \pm 1.96 \sqrt{\frac{\left(\frac{852}{1400}\right)\left(\frac{548}{1400}\right)}{1400}}$ *MIAIAI*

Note: Award *MI* for $\frac{852}{1400} \pm \dots$, *AI* for 1.96 and *AI* for $\sqrt{\frac{\left(\frac{852}{1400}\right)\left(\frac{548}{1400}\right)}{1400}}$

$= [0.583, 0.634]$

AIAI

N5

[5 marks]

continued ...

Question 3 continued

(b) (i) using the interval symmetry **MI**

$$p = \frac{0.2511 + 0.3155}{2} = \frac{0.5666}{2} = 0.2833 \quad \text{AI}$$

hence

$$n = 0.2833 \times 600 = 169.98$$

$$n = 170 \quad \text{AI}$$

(ii) $1 - p = q = 0.7167$ **(AI)**

$$\left(0.2833 + z_\alpha \times \sqrt{\frac{0.2833 \times 0.7167}{600}} \right) - \left(0.2833 + z_\alpha \times \sqrt{\frac{0.2833 \times 0.7167}{600}} \right) = 0.3155 - 0.2511$$

MI AI

$$z_\alpha = \frac{0.0322}{\sqrt{\frac{0.2833 \times 0.7167}{600}}} = 1.7504 \dots \approx 1.75 \quad \text{AI}$$

$$\alpha = 92.0 \% \text{ (do not penalize if only the } 92.0 \% \text{ is seen)} \quad \text{AI AI}$$

Note: Award first **AI** if 0.04 or 0.96 seen.

[9 marks]

continued ...

Question 3 continued

(c) H_0 : data can be modelled B(5, 0.35)

H_1 : data cannot be modelled B(5, 0.35)

A1

| | | | | | | |
|-------|-------|-------|-------|-------|------|------|
| x | 0 | 1 | 2 | 3 | 4 | 5 |
| f_o | 15 | 49 | 65 | 36 | 10 | 5 |
| f_e | 20.89 | 56.23 | 60.55 | 32.61 | 8.78 | 0.95 |

M1A2

Note: Award **A1** for 1 error, and **A0** for two or more errors.

| | | | | | |
|-------|-------|-------|-------|-------|--------|
| x | 0 | 1 | 2 | 3 | 4 or 5 |
| f_o | 15 | 49 | 65 | 36 | 15 |
| f_e | 20.89 | 56.23 | 60.55 | 32.61 | 9.72 |

R1

Note: **R1** is for combining 2 columns.

EITHER

$$\chi^2_{calc} = \frac{(15 - 20.89)^2}{20.89} + \frac{(49 - 56.23)^2}{56.23} + \dots + \frac{(15 - 9.72)^2}{9.72} \approx 6.14$$

(M1)A1

$$\chi^2_{5\%}(4) = 9.488, \nu = 4$$

A1A1

OR

by GDC with $\nu = 4$, $\chi^2_{GDC} \approx 6.14$, $p = 0.189$

(M1)A1A1A1

THEN

hence since $\chi^2_{5\%}(4) = 9.488 > \chi^2_{GDC}$ (or $p = 0.189 > 0.05$) then accept H_0

R1

[10 marks]

Total [24 marks]

4. (a) $ax \equiv b \pmod{p}$
 $\Rightarrow a^{p-2} \times ax \equiv a^{p-2} \times b \pmod{p}$ *MIAI*
 $\Rightarrow a^{p-1}x \equiv a^{p-2} \times b \pmod{p}$ *AI*
 but $a^{p-1} \equiv 1 \pmod{p}$ by Fermat's little theorem *RI*
 $\Rightarrow x \equiv a^{p-2} \times b \pmod{p}$ *AG*

Note: Award *MI* for some correct method and *AI* for correct statement.

[4 marks]

- (b) (i) $17x \equiv 14 \pmod{21}$
 $\Rightarrow x \equiv 17^{19} \times 14 \pmod{21}$ *MIAI*
 $17^6 \equiv 1 \pmod{21}$ *AI*
 $\Rightarrow x \equiv (1)^3 \times 17 \times 14 \pmod{21}$ *AI*
 $\Rightarrow x \equiv 7 \pmod{21}$ *AI*
- (ii) $x \equiv 7 \pmod{21}$
 $\Rightarrow x = 7 + 21t, t \in \mathbb{Z}$ *MIAI*
 $\Rightarrow 17(7 + 21t) + 21y = 14$ *AI*
 $\Rightarrow 119 + 357t + 21y = 14$
 $\Rightarrow 21y = -105 - 357t$ *AI*
 $\Rightarrow y = -5 - 17t$ *AI*

[10 marks]

Total [14 marks]

5. (a) **EITHER**

$$\begin{aligned} & \lim_{x \rightarrow 0} \left(\frac{1}{x} - \cot x \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{\tan x - x}{x \tan x} \right) && \text{M1A1} \\ &= \lim_{x \rightarrow 0} \left(\frac{\sec^2 x - 1}{x \sec^2 x + \tan x} \right), \text{ using l'Hopital} && \text{A1} \\ &= \lim_{x \rightarrow 0} \left(\frac{2 \sec^2 x \tan x}{2 \sec^2 x + 2x \sec^2 x \tan x} \right) && \text{A1A1} \\ &= 0 && \text{A1} \end{aligned}$$

OR

$$\begin{aligned} & \lim_{x \rightarrow 0} \left(\frac{1}{x} - \cot x \right) \\ &= \lim_{x \rightarrow 0} \left(\frac{\sin x - x \cos x}{x \sin x} \right) && \text{M1A1} \\ &= \lim_{x \rightarrow 0} \left(\frac{x \sin x}{\sin x + x \cos x} \right), \text{ using l'Hopital} && \text{A1} \\ &= \lim_{x \rightarrow 0} \left(\frac{\sin x + x \cos x}{2 \cos x - x \sin x} \right) && \text{A1A1} \\ &= 0 && \text{A1} \end{aligned}$$

[6 marks]

(b) $u_n = \frac{(x+2)^n}{3^n \times n}$ A1

$$\frac{u_{n+1}}{u_n} = \frac{(x+2)^{n+1}}{3^{n+1} \times (n+1)} \div \frac{(x+2)^n}{3^n \times n} = \frac{(x+2)n}{3(n+1)}$$
M1A1

$$\lim_{n \rightarrow \infty} \frac{(x+2)n}{3(n+1)} = \frac{(x+2)}{3}$$
M1A1

$$\left| \frac{(x+2)}{3} \right| < 1 \Rightarrow -5 < x < 1$$
M1A1

if $x = 1$ series is $1 + \frac{1}{2} + \frac{1}{3} + \dots$ which diverges A1

if $x = -5$ series is $-1 + \frac{1}{2} - \frac{1}{3} + \dots + \frac{(-1)^n}{n}$ which converges A1

hence interval is $-5 \leq x < 1$ A1

[10 marks]

continued ...

Question 5 continued

(c) (i) $f(x) = \ln(1 + \sin x)$, $f(0) = 0$ **AI**

$f'(x) = \frac{\cos x}{1 + \sin x}$, $f'(0) = 1$, **AI**

$f''(x) = \frac{-\sin x(1 + \sin x) - \cos^2 x}{(1 + \sin x)^2} = \frac{-(1 + \sin x)}{(1 + \sin x)^2} = \frac{-1}{1 + \sin x}$, $f''(0) = -1$ **AI**

$f'''(x) = \frac{\cos x}{(1 + \sin x)^2}$, $f'''(0) = 1$, **AI**

$\ln(1 + \sin x) \approx x - \frac{x^2}{2} + \frac{x^3}{6} - \dots$ **AI**

(ii) $-\sin x = \sin(-x)$ **MI**

so, $\ln(1 - \sin x) \approx -x - \frac{x^2}{2} - \frac{x^3}{6} - \dots$ **AI**

(iii) $\ln(1 + \sin x) - \ln(1 - \sin x)$
 $= \ln\left(\frac{1 + \sin x}{1 - \sin x}\right) \approx 2x + \frac{x^3}{3}$ **M1A1**

let $x = \frac{\pi}{6}$ then, $\ln\left(\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}}\right) = \ln 3 \approx 2\left(\frac{\pi}{6}\right) + \frac{\left(\frac{\pi}{6}\right)^3}{3}$ **M1A1A1**

$= \frac{\pi}{3}\left(1 + \frac{\pi^2}{216}\right)$ **AG**

[12 marks]

Total [28 marks]

6. (a) (i)

| | | | | | | |
|-------|---|---|---|---|---|---|
| $+_6$ | 0 | 1 | 2 | 3 | 4 | 5 |
| 0 | 0 | 1 | 2 | 3 | 4 | 5 |
| 1 | 1 | 2 | 3 | 4 | 5 | 0 |
| 2 | 2 | 3 | 4 | 5 | 0 | 1 |
| 3 | 3 | 4 | 5 | 0 | 1 | 2 |
| 4 | 4 | 5 | 0 | 1 | 2 | 3 |
| 5 | 5 | 0 | 1 | 2 | 3 | 4 |

the table is closed. A2
 the identity is 0 A1
 0 is in every row and column once so each element has a unique inverse A1
 addition is associative A1
 therefore $\{S, +_6\}$ is a group R1

(ii) $1+1+1+1+1+1=0$ M1
 $1+1+1+1+1=5$
 $1+1+1+1=4$
 $1+1+1=3$
 $1+1=2$
 so 1 is a generator of $\{S, +_6\}$ and the group is cyclic A1
 (since 5 is the additive inverse of 1) 5 is also a generator A1

(iii) $\{0, 2, 4\}$ A1

[11 marks]

(b) if a is a generator of group $(G, *)$ then so is a^{-1} A1
 if $(G, *)$ has exactly one generator a then $a = a^{-1}$ A1
 so $a^2 = e$ and $G = \{e, a\}$ A1R1
 so cyclic group with exactly one generator cannot have more than two elements AG

[4 marks]

(c) every element of a group has a unique inverse so Φ is a bijection A1
 $\Phi(ab) = (ab)^{-1} = b^{-1}a^{-1}$ M1A1
 if H is Abelian then it follows that
 $b^{-1}a^{-1} = a^{-1}b^{-1} = \Phi(a)\Phi(b)$ A1
 so Φ is an isomorphism R1
 if Φ is an isomorphism, then M1
 for all $a, b \in H$, $\Phi(ab) = \Phi(a)\Phi(b)$ M1
 $(ab)^{-1} = a^{-1}b^{-1}$
 $\Rightarrow b^{-1}a^{-1} = a^{-1}b^{-1}$ A1
 so H is Abelian R1

[9 marks]

Total [24 marks]